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A Note on Interpolating Bases

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It is shown that, given a compact metrizable Hausdorff space K and a dense sequence (t_n) in K, there is a monotone basis of $C(K)$ which is interpolating with nodes(t_n). This gives a positive answer to a question raised by Gurarii. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let X be a Banach space and (e_n) a sequence of elements of X. (e_n) is called a *basis* of X if, for each $x \in X$, there is exactly one representation of x of the form $x = \sum_{k=1}^{\infty} \alpha_k e_k$, α_k real or complex numbers. In this case, by uniform boundedness, we have $\sup_n \|S_n\| < \infty$, where $S_n(x) = \sum_{k=1}^{n} \alpha_k e_k$, $n = 1, 2, \ldots$. $\sup_n \|S_n\|$ is called the *basis constant* of (x_n). If $\sup_n \|S_n\| = 1$, then (e_n) is called a *monotone* basis.

In spaces of continuous functions on compact Hausdorff spaces K ($C(K)$-spaces), one can connect bases with the notion of interpolation of functions:

A basis (e_n) of a $C(K)$-space is called *interpolating* with nodes $t_n \in K$ if, for each $f \in C(K)$, $f(t_k) = S_n(f)(t_k)$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$ [5]. In this case the nodes (t_n) are necessarily dense in K [5, Proposition 1.3.7]. The foremost example of an interpolating basis is the Schauder system (e_n) in $C(0, 1)$. This basis is closely connected with a sequence of special peaked partitions of unity $e_{i,n}$, $i = 1, \ldots, n$, $n = 1, 2, \ldots$, of $C(0, 1)$. That is,

$$0 \leqslant e_{i,n} \leqslant 1, \quad \sum_{i=1}^{n} e_{i,n} = 1, \qquad e_{i,n}(t_k) = \begin{cases} 1, & i = k \\ 0, & i \neq k \quad \text{if } k = 1, \ldots, n, \end{cases}$$

and $\operatorname{span}\{e_{i,n+1}\}_{i=1}^{n+1} \supset \operatorname{span}\{e_{i,n}\}_{i=1}^{n}$. $\{t_1, t_2, \ldots\}$ is a given sequence dense in $[0, 1]$.. Here the Schaudersystem (with respect to $t_1, t_2, \ldots$) is defined by $e_n = e_{n,n}$, $n = 1, 2, \ldots$. One easily obtains that (e_n) is a monotone basis and, for each $f \in C(0, 1)$, $f(t_k) = \sum_{i=1}^{n} f(t_i)\, e_{i,n}(t_k) = S_n(f)(t_k)$, $k = 1, 2, \ldots, n$,

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$n=1,2,\ldots$ ([5, 2.3] broken-line interpolation). Note that the preceding conditions on $e_{i,n}$ imply that $\{e_{i,n} \mid i=1,\ldots,n\}$ corresponds to the unit vector basis in $\operatorname{span}\{e_{i,n}\}_{i=1}^{n} \cong l_{\infty}^{n}$ ($\|\sum_{i=1}^{n} \alpha_i e_{i,n}\| = \max_i |\alpha_i|$).

Gurarii showed in [1] that for each $\varepsilon>0$, any compact metrizable Hausdorff space K and sequence (t_n) dense in K there is a basis of $C(K)$ which has a basis constant $\leqslant 1+\varepsilon$ and which interpolates with nodes (t_n). Since the construction in [1] does not yield a monotone basis in general Gurarii raised the question whether there is, in any separable $C(K)$-space, a monotone basis interpolating for prescribed notes [1; 5, 4.3.5].

We give a positive answer to this question by constructing bases in a larger class of Banach spaces including $C(K)$-spaces, which are monotone and interpolate for given nodes. These bases again have the characteristic features of the Schauder systems described above. Indeed the Schauder systems on [0, 1] and their generalizations to functions with several variables [5, Chap. 3] are special cases of the following construction.

2. Admissible Bases in L_1-Predual Spaces

Let X be a Banach space whose dual is isometrically isomorphic to an L_1-space. For simplicity we consider Banach spaces over the real numbers, however, all constructions carry over to the complex field. The class of L_1-predual spaces of course includes $C(K)$-spaces and moreover, e.g., sublattices of $C(K)$, simplex spaces (i.e., spaces of continuous affine functions on a compact Choquet simplex) and $C_\sigma(K)$-spaces (i.e., where K is a compact Hausdorff space, $\sigma: K \to K$ is a continuous involution and $C_\sigma(K) = \{f \in C(K) \mid f(\sigma k) = -f(k) \text{ for all } k \in K\}$) [2].

Definition. Let X be an L_1-predual space and assume that Φ_n are elements of the extreme point set of the unit ball of X^*, ex $B(X^*)$. A basis (e_n) of X is called *interpolating* with nodes (Φ_n) if, for every $f \in X$, $\Phi_k(f) = \Phi_k(S_n(f))$, $k=1,\ldots,n$, $n=1,2,\ldots$. Here $S_n(f) = \sum_{k=1}^{n} \alpha_k e_k$ if $f = \sum_{k=1}^{\infty} \alpha_k e_k$. In case $X = C(K)$, the elements of ex $B(X^*)$ are the Dirac functionals (up to the sign) of the elements in K. Hence in this case the notion of an interpolating basis coincides with that of Section 1.

The following proposition is due to Lazar and Lindenstrauss [2, 4].

Proposition 1. *Let X be a separable Banach space. Then $X^* \cong L_1$ iff there are subspaces $E_1 \subset E_2 \subset \cdots \subset X$ such that $X = \overline{\bigcup E_n}$ and $E_n \cong l_\infty^n$ for all n.*

Let $e_{i,n}$, $i=1, \ldots, n$, be the unit vector basis of $E_n \cong l_\infty^n$. Then, after a suitable rearrangement of the indices, there are numbers $\alpha_{i,n}$ such that

$$e_{i,n} = e_{i,n+1} + \alpha_{i,n} e_{n+1,n+1}, \quad i=1, \ldots, n. \tag{*}$$

Put $e_n = e_{n,n}$ for all n. Then we have [4; 3, Lemma 1.1].

PROPOSITION 2. *(e_n) is a monotone basis of $X = \overline{\bigcup E_n}$.*

We call bases of X, which are constructed in this way, *admissible bases.* If X is a simplex space (which is equivalent to ex $B(X) \neq \emptyset$) then the $e_{i,n}$ can be taken to be peaked partions of unity (p.p.u.'s). That is, $0 \leqslant e_{i,n} \leqslant 1$ and $\sum_{i=1}^n e_{i,n} = 1$ in addition for all n [2]. In this case we call (e_n) a p.p.u. *basis* of X.

PROPOSITION 3. *Admissible bases are interpolating.*

Proof. We retain the preceding notation. For each n there is a unique $\Phi_n \in \text{ex } B(X^*)$ with

$$\Phi_n(e_{i,n+m}) = \begin{cases} 1, & i=n \\ 0, & i \neq n \end{cases} \quad \text{for all } i=1, \ldots, n+m, \ m=0, 1, 2, \ldots \quad [3, 6].$$

Using induction, one obtains, by (*),

$$S_n(f) = \sum_{i=1}^{n} \Phi_i(f)\, e_{i,n} \qquad \text{for each } f \in X.$$

Then clearly $\Phi_k(S_n(f)) = \Phi_k(f)$, $k=1, \ldots, n$. ∎

The Φ_n of the preceding proof are called the functionals *associated* with the admissible basis (e_n).

THEOREM. *Let X be a separable L_1-predual space. Assume that $\Phi_n \in \text{ex } B(X^*)$, $n=1, 2, \ldots,$ satisfy $\Phi_n \neq \pm \Phi_m$ if $n \neq m$ and $\overline{\{\pm \Phi_n \mid n=1, 2, \ldots\}} = \overline{\text{ex } B(X^*)}$ (ω^*-closures). Then there is an admissible basis $e_1, e_2, \ldots$ of X whose associated functionals are $\Phi_1, \Phi_2, \ldots$. Moreover, if X is a simplex space and $\Phi_n \in (\text{ex } B(X^*))_+$ then $e_1, e_2, \ldots$ can be chosen to be a* p.p.u.*-basis.*

If $X = C(K)$ this includes, in view of Propositions 2 and 3, a positive answer to the problem of Gurarii mentioned in Section 1. We postpone the proof of the Theorem to Section 4. First, we shall study some special properties of admissible bases.

3. Properties of Admissible Bases

Throughout this section let X be a separable L_1-predual space.

LEMMA 1 (Extension property). *Let $e_1, ..., e_n$ be an admissible basis of a subspace $E \subset X$. Assume that $G \subset X$ is finite dimensional and take $\varepsilon > 0$. Then there are a positive integer m and elements $e_{n+1}, ..., e_{n+m} \in X$ such that*

(1) *$e_1, ..., e_{n+m}$ is an admissible basis sequence spanning a subspace F*

(2) *$\inf\{\|g-f\| \mid f \in F\} \leqslant \varepsilon\|g\|$ for all $g \in G$.*

Proof. By [2, Theorem 3.1] there is an l_∞^{n+m}-subspace $F \subset X$ containing E with

$$\inf\{\|g-f\| \mid f \in F\} \leqslant \varepsilon\|g\| \qquad \text{for all } g \in G.$$

Since $E \cong l_\infty^n$ there are subspaces $E = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = F$, where $E_k \cong l_\infty^{n+k}$ for all k [4]. Let $e_{i,n}$, $i = 1, ..., n$, be the unit vector basis of $E \cong l_\infty^n$ such that $e_n = e_{n,n}$. Then one can find an arrangement of the indices of the unit vector basis of E_1, $e_{i,n+1}$, $i = 1, ..., n+1$, such that

$$e_{i,n} = e_{i,n+1} + \alpha_i e_{n+1,n+1}, \qquad i = 1, ..., n,$$

for some numbers α_i [4]. Put $e_{n+1} = e_{n+1,n+1}$. Induction concludes the proof. ▮

LEMMA 2 (Permutation property). *Let (e_k) be an admissible basis of X whose associated functions are (Φ_k). If $\Phi_{n+1}(e_n) = 0$ for some n then $e_1, ..., e_{n-1}, e_{n+1}, e_n, e_{n+2}, ...$ is an admissible basis of X with associated functionals $\Phi_1, ..., \Phi_{n-1}, \Phi_{n+1}, \Phi_n, \Phi_{n+2}, ...$*

Proof. Put $E_k = \operatorname{span}\{e_1, ..., e_k\}$. Hence $E_k = l_\infty^k$, $k = 1, 2, ...$ Consider the unit vector basis $e_{i,k}$ of E_k, $i = 1, ..., k$, such that $e_k = e_{k,k}$. We have

(1) $e_{i,n-1} = e_{i,n} + \Phi_n(e_{i,n-1}) e_n$, $i = 1, ..., n-1$ and

(2) $e_{j,n} = e_{j,n+1} + \Phi_{n+1}(e_{j,n}) e_{n+1}$, $j = 1, ..., n$. Hence

(3) $e_{n,n+1} = e_n$ by our assumption. We obtain

(4) $e_{i,n-1} = e_{i,n+1} + \Phi_n(e_{i,n-1}) e_{n,n+1} + (\Phi_{n+1}(e_{i,n}) + \Phi_n(e_{i,n-1}) \Phi_{n+1}(e_n)) e_{n+1} = e_{i,n+1} + \Phi_n(e_{i,n-1}) e_{n,n+1} + \Phi_{n+1}(e_{i,n}) e_{n+1}$. Put

(5) $\tilde{e}_{i,n} = e_{i,n+1} + \Phi_n(e_{i,n-1}) e_{n,n+1}$, $i = 1, ..., n-1$, $\tilde{e}_{n,n} = e_{n+1}$. Then $\tilde{e}_{1,n}, ..., \tilde{e}_{n,n}$ is the unit vector basis of l_∞^n since

$$\left\|\sum_{i=1}^n \lambda_i \tilde{e}_{i,n}\right\| = \sup\left\{\left|\Phi_j\left(\sum_{i=1}^n \lambda_i \tilde{e}_{i,n}\right)\right| \,\middle|\, j \in \{1, ..., n-1, n+1\}\right\}$$

$$= \max_{i \leqslant n} |\lambda_i| \qquad \text{for all } \lambda_i.$$

Moreover, put

(6) $\tilde{e}_{i,n+1}=e_{i,n+1}$, $i=1, ..., n-1$, $\tilde{e}_{n,n+1}=e_{n+1}$, $\tilde{e}_{n+1,n+1}=e_{n,n+1}$.

Again, $\|\sum_{j=1}^{n+1}\lambda_j\tilde{e}_{j,n+1}\| = \sup\{|\Phi_k(\sum_{j=1}^{n+1}\lambda_j\tilde{e}_{j,n+1})| \mid k=1, ..., n+1\} = \max_{j\leqslant n+1}|\lambda_j|$, for all λ_j. We have (1, 4, 5)

$$e_{i,n-1}=\tilde{e}_{i,n}+\Phi_{n+1}(e_{i,n})\,\tilde{e}_{n,n}, \qquad i=1, ..., n-1$$

and

$$\tilde{e}_{i,n}=\tilde{e}_{i,n+1}+\Phi_n(e_{i,n-1})\,\tilde{e}_{n+1,n+1}, \qquad i=1, ..., n-1 \quad ((3), (4), (5), (6)),$$
$$\tilde{e}_{n,n}=\tilde{e}_{n,n+1}.$$

Since $\tilde{e}_{n,n}=e_{n+1}$, $\tilde{e}_{n+1,n+1}=e_n$ we obtain Lemma 2. ∎

In the following lemma let $e_{i,n}$, $i=1, ..., n$, be the unit vector basis of $\operatorname{span}\{e_1, ..., e_n\}$ with $e_n=e_{n,n}$, $n=1, 2, ...$.

LEMMA 3 (Exchange property). *Let (e_k) be an admissible basis of X whose associated functionals are (Φ_k). Then, for any index set $\{n+1, ..., n+m\}$, the sequence $e_1, ..., e_n$, $e_{n+1,n+m}$, $e_{n+2,n+m}, ..., e_{n+m}$, $e_{n+m+1}, ...$ is an admissible basis of X whose associated functionals are again (Φ_k) (in the same order).*

Proof. It suffices to assume $m=2$, Lemma 3 follows then by using induction. (The case $m=1$ is trivial because $e_{n+1,n+1}=e_{n+1}$). We have

$$\begin{aligned} e_{i,n}&=e_{i,n+1}+\Phi_{n+1}(e_{i,n})\,e_{n+1,n+1}\\ &=e_{i,n+2}+\Phi_{n+1}(e_{i,n})\,e_{n+1,n+2}+(\Phi_{n+2}(e_{i,n+1})\\ &\quad+\Phi_{n+1}(e_{i,n})\,\Phi_{n+2}(e_{n+1,n+1}))\,e_{n+2,n+2}, \qquad i=1, ..., n. \quad (1)\end{aligned}$$

Put

$$\begin{aligned}\tilde{e}_{i,n+1}&=e_{i,n+2}+(\Phi_{n+2}(e_{i,n+1})\\ &\quad+\Phi_{n+1}(e_{i,n})\,\Phi_{n+2}(e_{n+1,n+1}))\,e_{n+2,n+2} \qquad i=1, ..., n, \quad (2)\end{aligned}$$

and

$$\tilde{e}_{n+1,n+1}=e_{n+1,n+2}. \qquad (3)$$

Then $\{\tilde{e}_{i,n+1} \mid i=1, ..., n+1\}$ is the unit vector basis of l_∞^{n+1}. Moreover, by (1), (2), (3), $e_{i,n}=\tilde{e}_{i,n+1}+\Phi_{n+1}(e_{i,n})\,\tilde{e}_{n+1,n+1}$, $i=1, ..., n$, and $\tilde{e}_{j,n+1}=e_{j,n+2}+\Phi_{n+2}(\tilde{e}_{j,n+1})\,e_{n+2,n+2}$, $j=1, ..., n+1$. The latter equation follows from (3), if $j=n+1$, since $\Phi_{n+2}(e_{n+1,n+2})=0$. If $j<n+1$ this equation follows from (2) since then

$$\Phi_{n+2}(\tilde{e}_{j,n+1})=\Phi_{n+2}(e_{i,n+1})+\Phi_{n+1}(e_{i,n})\,\Phi_{n+2}(e_{n+1,n+1}).$$

This concludes the proof in view of (3). ∎

In the proof of the next lemma we make use of the following fact due to Lazar and Lindenstrauss [2, Theorem 2.1], we state here only a special version:

> *Consider $\Gamma=\operatorname{conv}(\Gamma_1\cup-\Gamma_1)$, where Γ_1 is a ω^*-closed face of $B(X^*)$. Let $f\colon B(X^*)\to\mathbb{R}$ be concave, ω^*-continuous and assume $f(x^*)+f(-x^*)\geqslant 0$ for all $x^*\in B(X^*)$. Suppose $\hat{e}\colon\Gamma\to\mathbb{R}$ is ω^*-continuous, affine and $\hat{e}(0)=0$ such that $\hat{e}\leqslant f_{|\Gamma}$. Then there is $e\in X$ with $\hat{e}(x^*)=x^*(e)$, $x^*\in\Gamma$, and $x^*(e)\leqslant f(x^*)$, $x^*\in B(X^*)$.*

LEMMA 4 (Dual extension property). *Let $e_1,\dots,e_n$ be an admissible basis of a subspace $E\subset X$. Let $\Phi_1,\dots,\Phi_n\in\operatorname{ex}B(X^*)$ be such that $\Phi_{k|E}$, $k=1,\dots,n$, are the corresponding associated functionals. Consider $\Phi\in\operatorname{ex}B(X^*)\backslash\{\pm\Phi_k\,|\,k=1,\dots,n\}$. Then there is $e\in X$ such that $e_1,\dots,e_n$, e is an admissible basic sequence spanning a subspace $F\subset X$ and $\Phi_{1|F},\dots,\Phi_{n|F}$, $\Phi_{|F}$ are the corresponding associated functionals.*

Proof. Let $e_{i,n}$, $i=1,\dots,n$, be the unit vector basis of $l^n_\infty\cong E$ with $e_n=e_{n,n}$. Put $\Gamma_1=\operatorname{conv}\{\Phi_1,\dots,\Phi_n,\Phi\}$ and $\Gamma=\operatorname{conv}(\Gamma_1\cup-\Gamma_1)$. Then Γ_1 is a ω^*-closed face of $B(X^*)$ (because $X^*\cong L_1$ and $\Phi_1,\dots,\Phi_n$, $\Phi\in\operatorname{ex}B(X^*)$). Define $f\colon B(X^*)\to\mathbb{R}$ by

$$f(x^*)=\min\left\{\left(1-\sum_{i=1}^{n}\Theta_i x^*(e_{i,n})\right)\Big/\left(1-\sum_{i=1}^{n}\Theta_i\Phi(e_{i,n})\right)\right|$$

$$\Theta_i=\pm 1,\qquad i=1,\dots,n,\quad\text{such that}\ \sum_{i=1}^{n}\Theta_i\Phi(e_{i,n})\neq 1\Bigg\},\qquad x^*\in B(X^*).$$

It is easily checked that f is concave, ω^*-continuous and $f(x^*)+f(-x^*)\geqslant 0$ for all $x^*\in B(X^*)$. We even have $f(x^*), f(-x^*)\geqslant 0$ since $\sum_{i=1}^n|x^*(e_{i,n})|\leqslant 1$. Put

(1) $\hat{e}(\lambda\Phi+\sum_{i=1}^n\lambda_i\Phi_i)=\lambda$, if $|\lambda|+\sum_{i=1}^n|\lambda_i|\leqslant 1$. We have $\hat{e}(x^*)\leqslant f(x^*)$ for all $x^*\in\Gamma$. Hence there is $e\in X$ such that

(2) $x^*(e)=\hat{e}(x^*)$, $x^*\in\Gamma$,

(3) $x^*(e)\leqslant f(x^*)$, $x^*\in B(X^*)$. Put

(4) $e_{i,n+1}=e_{i,n}-\Phi(e_{i,n})\,e$, $i=1,\dots,n$, and $e_{n+1,n+1}=e$.

Then by (1), (2), (4),

$$\Phi_j(e_{i,n+1}) = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}, \qquad \Phi(e_{i,n+1}) = 0,\ i<n+1,$$

$$\Phi(e_{n+1,n+1}) = 1, \qquad \Phi_j(e_{n+1,n+1}) = 0, \quad j=1, ..., n.$$

Moreover, by (3) and (4),

$$\sum_{i=1}^{n+1} |x^*(e_{i,n+1})| \leqslant 1 \qquad \text{for all} \quad x^* \in B(X^*).$$

This proves that $e_{i,n+1}$, $i=1, ..., n+1$, is the unit vector basis of l_∞^{n+1}. By definition (4) we have

$$e_{i,n} = e_{i,n+1} + \Phi(e_{i,n})\, e, \qquad i=1, ..., n.$$

This proves Lemma 4. ∎

LEMMA 5 (Perturbation property I). *Let* $e_1, ..., e_n, e_{n+1}$ *be an admissible basis of a subspace* $E \subset X$ *and consider* $\Phi_1, ..., \Phi_n, \Phi_{n+1} \in \operatorname{ex} B(X^*)$ *such that* $\Phi_{1|E}, ..., \Phi_{n+1|E}$ *are the corresponding associated functionals. Then for any* $\varepsilon > 0$ *there is a* $\delta > 0$ *(depending on E and* ε*) satisfying the following*:

For any $\Phi \in \operatorname{ex} B(X^*)$ *with*

(1) $\|\Phi_{|E} - \Phi_{n+1|E}\| \leqslant \delta$

there is $e \in X$ *such that*

(2) $e_1, ..., e_n$, e *is an admissible basis of a subspace* $F \subset X$ *whose associated functionals are* $\Phi_{1|F}, ..., \Phi_{n|F}, \Phi_{|F}$,

(3) $\|e - e_{n+1}\| \leqslant \varepsilon$

Proof. Let $e_{i,n}$, $i=1, ..., n$, be the unit vector basis of $\operatorname{span}\{e_1, ..., e_n\} \cong l_\infty^n$ with $e_n = e_{n,n}$. Fix $\varepsilon > 0$. A continuity argument yields $\delta > 0$ with $\delta < \min(1, \varepsilon)$ satisfying

$$0 < 1 - \sum_{i=1}^{n} \Theta_i x^*(e_{i,n})$$

and

(4) $|(1 - \sum_{i=1}^n \Theta_i x^*(e_{i,n}))^{-1} - (1 - \sum_{i=1}^n \Theta_i \Phi_{n+1}(e_{i,n}))^{-1}| \leqslant \varepsilon/2$ for all $\Theta_i = \pm 1$, $i=1, ..., n$, with $\sum_{i=1}^n \Theta_i \Phi_{n+1}(e_{i,n}) \neq 1$,

whenever $\|x^*_{|E} - \Phi_{n+1|E}\| \leqslant \delta$, $x^* \in B(X^*)$. (Continuity of the functions $x^*_{|E} \to \sum_{i=1}^n \Theta_i x^*(e_{i,n})$.)

Now, fix $\Phi \in \operatorname{ex} B(X^*)$ with $\|\Phi_{|E} - \Phi_{n+1|E}\| \leqslant \delta$. Since $1 > \delta$ we have $\Phi \notin \{\pm\Phi_1, ..., \pm\Phi_n, -\Phi_{n+1}\}$. We may assume $\Phi \neq \Phi_{n+1}$, otherwise there is nothing to prove. Define, for $x^* \in B(X^*)$,

$$f(x^*) = \min \left\{ \left(1 - \sum_{i=1}^{n} \Theta_i x^*(e_{i,n}) \right) \Big/ \left(1 - \sum_{i=1}^{n} \Theta_i \Phi(e_{i,n}) \right) \right| \\ \Theta_i = \pm 1,\ i = 1, ..., n, \text{ such that } \sum_{i=1}^{n} \Theta_i \Phi_{n+1}(e_{i,n}) \neq 1 \Bigg\},$$

$$g(x^*) = \varepsilon + x^*(e_{n+1}), \qquad h(x^*) = \min(f(x^*), g(x^*)).$$

Clearly, f, g, h are ω^*-continuous and concave. Furthermore, we have

(5) $f(x^*), f(-x^*) \geqslant 0$, $g(x^*) + g(-x^*) \geqslant 0$ for all $x^* \in B(X^*)$.

We claim

(6) $h(x^*) + h(-x^*) \geqslant 0$ for all $x^* \in B(X^*)$.

To prove (6), in view of (5), we only have to check the case $h(x^*) = f(x^*)$, $h(-x^*) = g(-x^*)$. Note, $x^*_{|E} = \sum_{i=1}^{n+1} \lambda_i \Phi_{i|E}$ for some λ_i such that $\sum_{i=1}^{n+1} |\lambda_i| \leqslant 1$. Hence $x^*_{|E} = (1-\lambda)\, y^*_{|E} + \lambda z^*_{|E}$ for some $0 \leqslant \lambda \leqslant 1$, where $y^*(e_{n+1}) = 0$, $z^*_{|E} \in \{\pm\Phi_{n+1|E}\}$. By definition we obtain

(7) $g(-x^*) = \varepsilon - \lambda z^*(e_{n+1})$. Using concavity and (5) we conclude

(8) $f(x^*) \geqslant (1-\lambda) f(y^*) + \lambda f(z^*) \geqslant \lambda f(z^*)$.

Hence by (4),

$$\begin{aligned} &h(x^*) + h(-x^*) \\ &\quad = g(-x^*) + f(x^*) \\ &\quad \geqslant \varepsilon - \lambda z^*(e_{n+1}) + \lambda \left(1 - z^* \left(\sum_{i=1}^{n} \Theta_i e_{i,n} \right) \right) \Big/ \left(1 - \Phi \left(\sum_{i=1}^{n} \Theta_i e_{i,n} \right) \right) \\ &\quad \geqslant \varepsilon - \lambda z^*(e_{n+1}) - \lambda\varepsilon + \lambda \left(1 - z^* \left(\sum_{i=1}^{n} \Theta_i e_{i,n} \right) \right) \Big/ \left(1 - \Phi_{n+1} \left(\sum_{i=1}^{n} \Theta_i e_{i,n} \right) \right) \end{aligned}$$

for a suitable choice of signs Θ_i. (We used $|1 - z^*(\sum_{i=1}^{n} \Theta_i e_{i,n})| \leqslant 2$).

There are two possibilities:

$$\text{Either } z^* = \Phi_{n+1} \qquad \text{or} \qquad z^* = -\Phi_{n+1}.$$

If $z^* = \Phi_{n+1}$, then

$$h(x^*) + h(-x^*) \geqslant \varepsilon - \lambda - \lambda\varepsilon + \lambda \geqslant 0.$$

If $z^* = -\Phi_{n+1}$, then

$$h(x^*) + h(-x^*) \geqslant \varepsilon + \lambda - \lambda\varepsilon \geqslant 0$$

since in any case

$$\left(1 - z^*\left(\sum_{i=1}^{n} \Theta_i e_{i,n}\right)\right) \Big/ \left(1 - \Phi_{n+1}\left(\sum_{i=1}^{n} \Theta_i e_{i,n}\right)\right) \geqslant 0.$$

This proves the claim.

Now we proceed as in the proof of Lemma 4. Put $\Gamma = \operatorname{conv}(\{\pm\Phi_1, ..., \pm\Phi_n, \pm\Phi\})$ and define $\hat{e}(\lambda\Phi + \sum_{i=1}^{n} \lambda_i \Phi_i) = \lambda$. Since $\|\Phi_{|E} - \Phi_{n+1|E}\| \leqslant \delta \leqslant \varepsilon$ we obtain $|\hat{e}(x^*) - x^*(e_{n+1})| \leqslant \varepsilon$ for all $x^* \in \Gamma$. Hence $\hat{e}(x^*) \leqslant g(x^*)$ for all $x^* \in \Gamma$, and, since $\hat{e}(x^*) \leqslant f(x^*)$, $\hat{e}(x^*) \leqslant h(x^*)$ for all $x^* \in \Gamma$. According to [2, Theorem 2.1] there is an $e \in X$ with $\hat{e}(x^*) = x^*(e)$ for all $x^* \in \Gamma$ and $x^*(e) \leqslant h(x^*)$ for all $x^* \in B(X^*)$.

This implies $|x^*(e) - x^*(e_{n+1})| \leqslant \varepsilon$ for all $x^* \in B(X^*)$ which yields (3). Moreover, put $e_{i,n+1} = e_{i,n} - \Phi(e_{i,n})\, e$, $i = 1, ..., n$, and $e_{n+1,n+1} = e$. In view of $x^*(e) \leqslant f(x^*)$ for all $x^* \in B(X^*)$, $e_{i,n+1}$, $i = 1, ..., n+1$, must be the unit vector basis of l_∞^{n+1} (this is the same argument as in the last part of the proof of Lemma 4). This concludes the proof, since then

$$e_{i,n} = e_{i,n+1} + \Phi(e_{i,n})\, e_{n+1,n+1} \qquad \text{and} \qquad e_{n+1,n+1} = e. \quad \blacksquare$$

LEMMA 6 (Perturbation property II). *Let Δ be a ω^*-dense subset of* ex $B(X^*)$. *Consider an admissible basis $e_1, ..., e_n, e_{n+1}, ..., e_{n+m}$ of a subspace E of X and assume $\Phi_1, ..., \Phi_{m+n} \in$ ex $B(X^*)$ are such that $\Phi_{1|E}, ..., \Phi_{n+m|E}$ are the corresponding associated functionals. Then for any $\varepsilon > 0$ there are $\psi_{k_1}, ..., \psi_{k_m} \in \Delta$ and $f_1, ..., f_m \in X$ such that*

(1) *$e_1, ..., e_n, f_1, ..., f_m$ is an admissible basis of a subspace F of X whose associated functionals are $\Phi_{1|F}, ..., \Phi_{n|F}, \psi_{k_1|F}, ..., \psi_{k_m|F}$,*

(2) *$\inf\{\|e - f\| \mid f \in F\} \leqslant \varepsilon\|e\|$ for all $e \in E$.*

Proof. We use induction on m. The case $m = 1$ is proven by Lemma 5. Assume Lemma 6 holds for a fixed $m - 1$. Then let $e_1, ..., e_n, e_{n+1}, ..., e_{n+m}$, $\Phi_1, ..., \Phi_{n+m}$ be as in the hypothesis of Lemma 6. The induction hypothesis yields $f_1, ..., f_{m-1}, \psi_{k_1}, ..., \psi_{k_{m-1}} \in \Delta$ such that

(3) $e_1, ..., e_{n+1} f_1, ..., f_{m-1}$ is an admissible basis of a subspace $G \subset X$, whose associated functionals are $\Phi_{1|G}, ..., \Phi_{n+1|G}, \psi_{k_1|G}, ..., \psi_{k_{m-1}|G}$.

(4) $\inf\{\|e - g\| \mid g \in G\} \leqslant \varepsilon\|e\|/3$ for all $e \in E$.

Apply Lemma 3 to obtain $g_0, ..., g_{m-1}$ such that $e_1, ..., e_n, g_0, ..., g_{m-1}$ again is an admissible basis of G with the same associated functionals as before and such that in addition

$$\psi_{k_j}(g_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \qquad i, j = 0, 1, ..., m-1,$$

where $\psi_{k_0} = \Phi_{n+1}$. Using Lemma 2 and induction we see that $e_1, ..., e_n$, $g_1, ..., g_{m-1}$, g_0 is an admissible basis with associated functionals $\Phi_{1|G}, ..., \Phi_{1|G}$, $\psi_{k_{1|G}}, ..., \psi_{k_{m-1|G}}$, $\Phi_{n+1|G}$. Apply Lemma 5 again to find $f \in X$, $\psi_{k_m} \in \Delta$, such that

(5) $e_1, ..., e_n$, $g_1, ..., g_{m-1}$, f is an admissible basis of a subspace $F \subset X$ with associated functionals $\Phi_{1|F}, ..., \Phi_{n|F}$, $\psi_{k_{1|F}}, ..., \psi_{k_{m-1|F}}$, $\psi_{k_{m|F}}$,

(6) $\inf\{\|g-f\| \mid f \in F\} \leqslant \varepsilon \|g\|/3$ for all $g \in G$.

Arguments (4)–(6) yield (1) and (2) and conclude the induction. ∎

4. Proof of the Theorem

Let $\Phi_n \in \text{ex } B(X^*)$ be such that $\{\pm\Phi_n \mid n \in \mathbb{N}\}$ is ω^*-dense in ex $B(X^*)$ and assume that for each m we have

(1) $\Phi_m \notin \{\pm\Phi_n \mid n \neq m, n = 1, 2, ...\}$.

Fix a norm dense subset $\{x_k \mid k = 1, 2, ...\}$ of $B(X)$. We use induction to find an admissible basis (e_n) of X whose associated functionals are $\Phi_1, \Phi_2, \Phi_3, ...$ For $m_0 = 1$ one can take any norm one vector $e_1 \in X$ such that $\Phi_1(e_1) = 1$. Now assume that we have an admissible basic sequence $e_1, ..., e_{m_n} \in X$ already such that $\Phi_{1|E}, ..., \Phi_{m_n|E}$ are the corresponding associated functionals ($E = \text{span}\{e_1, ..., e_{m_n}\}$). Put

(2) $G = \text{span}\{x_1, ..., x_n\}$.

Apply Lemma 1 to find $\tilde{f}_1, ..., \tilde{f}_r \in X$ such that $e_1, ..., e_{m_n}$, $\tilde{f}_1, ..., \tilde{f}_r$ is an admissible basis sequence spanning a subspace $\tilde{F}$ with

(3) $\inf\{\|g-\tilde{f}\| \mid \tilde{f} \in \tilde{F}\} \leqslant \|g\|/n$ for all $g \in G$. Assume $\psi_1, ..., \psi_r \in$ ex $B(X^*)$ are such that $\Phi_{1|\tilde{F}}, ..., \Phi_{m_n|\tilde{F}}$, $\psi_{1|F}, ..., \psi_{r|\tilde{F}}$ are the corresponding associated functionals. Then apply Lemma 6 to find $f_1, ..., f_r$ and $\Phi_{k_1}, ..., \Phi_{k_r}$, $k_1, ..., k_r > m_n$, such that

(4) $e_1, ..., e_{m_n}$, $f_1, ..., f_r$ is an admissible basis sequence spanning a subspace F of X whose associated functionals are $\Phi_{1|F}, ..., \Phi_{m_n|F}$, $\Phi_{k_{1|F}}, ..., \Phi_{k_{r|F}}$ and such that

(5) $\inf\{\|f-\tilde{f}\| \mid f \in F\} \leqslant \|\tilde{f}\|/n$ for all $\tilde{f} \in \tilde{F}$.

It is possible to apply Lemma 6 in this situation, we may at first get such f_j where $-\Phi_{k_{j|F}}$ are the associated functionals for some j. In this case we take $-f_j$ instead f_j. Hence without loss of generality we can assume there is such an admissible basis sequence satisfying (4) and (5).

Note that $k_1, ..., k_r$ may not be ordered according to the order of the integers. Furthermore, there may be "gaps" between the k_j. To fill in the gaps, put $s = (\max_{j=1,\dots,r} k_j) - m_n$ and consider

(6) $N = \{m_n + 1, m_n + 2, \dots, m_n + s\} \setminus \{k_1, \dots, k_r\}$.

Put $N = \{k_{r+1}, k_{r+2}, \dots, k_s\}$. Apply Lemma 4, $s - r$ times to find $f_{r+1}, \dots, f_s$ such that $e_1, \dots, e_{m_n}, f_1, \dots, f_r, f_{r+1}, \dots, f_s$ is an admissible basis sequence spanning a subspace $H \subset X$ whose associated functionals are

$$\Phi_{1|_H}, \dots, \Phi_{m_n|_H}, \qquad \Phi_{k_1|_H}, \dots, \Phi_{k_r|_H}, \qquad \Phi_{k_{r+1}|_H}, \dots, \Phi_{k_s|_H}.$$

Of course, (5) remains valid for H instead of F since $F \subset H$. Finally, we reorder the Φ_{k_j}. To this end apply Lemma 3 to find $g_1, \dots, g_s \in H$ such that $e_1, \dots, e_{m_n}, g_1, \dots, g_s$ is an admissible basis of H whose associated functionals are the same as before and such that $\Phi_{k_j}(g_i) = 0$ if $i \neq j$, $i, j = 1, \dots, s$. By Lemma 2 one may permute the indices of the g_j. Hence let $h_1, \dots, h_s$ be a permutation of $1, \dots, s$ such that

$$k_{h_1} = m_n + 1, \qquad k_{h_2} = m_n + 2, \dots, k_{h_s} = m_n + s.$$

Put $e_{m_n+j} = g_{h_j}$, $j = 1, \dots, s$. Then, $e_1, \dots, e_{m_n}, e_{m_n+1}, \dots, e_{m_n+s}$ is an admissible basis of H whose associated functionals are

$$\Phi_{1|_H}, \dots, \Phi_{m_n|_H}, \qquad \Phi_{m_n+1|_H}, \qquad \Phi_{m_n+2|_H}, \dots, \Phi_{m_n+s|_H}.$$

In view of (3) and (5) we have

(7) $\inf\{\|g - h\| \mid h \in H\} \leqslant 2\|g\|/n$ for all $g \in G$.

Put $m_{n+1} = m_n + s$. This finishes the induction, which clearly yields an interpolating admissible basic sequence whose associated functionals are $\Phi_1, \Phi_2, \dots$. By (2) and (7) this basic sequence is a basis of X. In case X is a simplex space we can start with $e_1 = 1$. Then, if $\Phi_n \in \operatorname{ex} B(X^*)_+$ (i.e., Φ_n are the Dirac functionals of extreme points of the underlying simplex), we have $\Phi_n(e_1) = 1$ for all n. If $e_{i,n}$, $i = 1, \dots, n$, is the unit vector basis of $\operatorname{span}\{e_1, \dots, e_n\}$ then one obtains by induction $0 \leqslant e_{i,n} \leqslant e_1 = 1$ and $\sum_{i=1}^n e_{i,n} = 1$. Hence in this case (e_n) is a p.p.u.-basis. ▮

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